

THE STRUCTURE OF CORINGS.

INDUCTION FUNCTORS, MASCHKE-TYPE THEOREM, AND FROBENIUS AND GALOIS-TYPE PROPERTIES.

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ABSTRACT. Given a ring A and an A -coring \mathcal{C} we study when the forgetful functor from the category of right \mathcal{C} -comodules to the category of right A -modules and its right adjoint $- \otimes_A \mathcal{C}$ are separable. We then proceed to study when the induction functor $- \otimes_A \mathcal{C}$ is also the left adjoint of the forgetful functor. This question is closely related to the problem when $A \rightarrow {}_A\mathrm{Hom}(\mathcal{C}, A)$ is a Frobenius extension. We introduce the notion of a Galois coring and analyse when the tensor functor over the subring of A fixed under the coaction of \mathcal{C} is an equivalence. We also comment on possible dualisation of the notion of a coring.

1. INTRODUCTION

A coring is a generalisation of a coalgebra introduced by M. Sweedler in [27]. It has been recently pointed out by M. Takeuchi [28] that new examples of corings can be provided by *entwining structures* introduced in [7] in the context of gauge theory on non-commutative spaces. Entwining structures and modules associated to them generalise the notion of Doi-Koppinen Hopf modules introduced in [14] [19]. Various structure theorems concerning Doi-Koppinen modules can be formulated more generally in terms of entwined modules (see recent paper [9] or [3] for an exhaustive list of references). In the present paper we argue that many of those structure theorems are in fact special cases of structure theorems for the category of comodules of a coring.

We begin in Section 2 by recalling the definition of a coring and listing examples of corings coming from entwining structures and their recent generalisation [9] motivated by Doi-Koppinen modules for a weak Hopf algebra introduced in [1]. In Section 3 we study when the forgetful functor from the category of right comodules of an A -coring \mathcal{C} to the category of right A -modules is separable. This turns out

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to be equivalent to coseparability of \mathcal{C} and implies a Maschke-type theorem for a coring. We also study when the induction functor $- \otimes_A \mathcal{C}$ which is the right adjoint of the forgetful functor above is separable. In Section 4 we derive conditions for the induction functor to be the left adjoint of the forgetful functor as well. Put differently, since a functor which has the same left and right adjoint is known as a *Frobenius functor* [12], we derive in Section 4 conditions for the forgetful functor (or, equivalently, the induction functor) to be a Frobenius functor. This problem is closely related to the question when $A \rightarrow {}_A \text{Hom}(\mathcal{C}, A)$ is a Frobenius extension. Such a question is particularly relevant in the present context as it is known that there is a close relationship between Frobenius extensions $A \rightarrow R$ and certain A -coring structures on R [10] [17]. Next in Section 5 we analyse the situation in which the ring A is itself a right \mathcal{C} -comodule. We define the subring of coinvariants B of A and study the induction functor $- \otimes_B A$ from the category of right B -modules to the category of right \mathcal{C} -comodules. Motivated by coalgebra-Galois extensions we introduce the notion of a Galois coring and show that if \mathcal{C} is a flat left A -module then $- \otimes_B A$ is an equivalence if and only if A is a faithfully flat left B -module and \mathcal{C} is Galois. Finally, in Section 6 we define a \mathcal{C} -algebroid or a \mathcal{C} -ring as a dualisation of A -coring and gather some comments on dualisation of the results of the previous sections to the case of \mathcal{C} -algebroids.

We use the following conventions. For an object V in a category, the identity morphism $V \rightarrow V$ is denoted by V . All rings in this paper have 1, a ring map is assumed to respect 1, and unless stated otherwise all modules over a ring are assumed to be unital. For a ring A , \mathbf{M}_A (resp. ${}_A \mathbf{M}$) denotes the category of right (resp. left) A -modules. The morphisms in this category are denoted by $\text{Hom}_A(M, N)$ (resp. ${}_A \text{Hom}(M, N)$). The action of A is denoted by a dot between elements. If A, B are rings and M, N are (A, B) -bimodules then ${}_A \text{Hom}_B(M, N)$ denotes the set of (A, B) -bimodule maps. For an (A, A) -bimodule M , M^A denotes the subbimodule of invariants, i.e., $M^A = \{m \in M \mid \forall a \in A, a \cdot m = m \cdot a\}$.

Throughout the paper k denotes a commutative ring with unit. We assume that all the algebras are over k and unital, and coalgebras are over k and counital. Unadorned tensor product is over k . For a k -algebra A we use μ to denote the

product as a map and 1 to denote unit both as an element of A and as a map $k \rightarrow A$, $\alpha \rightarrow \alpha 1$.

For a k -coalgebra C we use Δ to denote the coproduct and ϵ to denote the counit. Notation for comodules is similar to that for modules but with subscripts replaced by superscripts, i.e. \mathbf{M}^C is the category of right C -comodules, ρ^M is a right coaction, ${}^C\mathbf{M}$ is the category of left C -comodules and ${}^M\rho$ is a left coaction etc. We use the Sweedler notation for coproducts and coactions, i.e. $\Delta(c) = c_{(1)} \otimes c_{(2)}$, $\rho^M(m) = m_{(0)} \otimes m_{(1)}$ (summation understood). We use similar notation for corings.

2. NEW EXAMPLES OF CORINGS

Let A be a ring. Recall from [27] that an A -coring is an (A, A) -bimodule \mathcal{C} together with (A, A) -bimodule maps $\Delta_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C}$ called a coproduct and $\epsilon_{\mathcal{C}} : \mathcal{C} \rightarrow A$ called a counit, such that

$$(\Delta_{\mathcal{C}} \otimes_A \mathcal{C}) \circ \Delta_{\mathcal{C}} = (\mathcal{C} \otimes_A \Delta_{\mathcal{C}}) \circ \Delta_{\mathcal{C}}, \quad (\epsilon_{\mathcal{C}} \otimes_A \mathcal{C}) \circ \Delta_{\mathcal{C}} = (\mathcal{C} \otimes_A \epsilon_{\mathcal{C}}) \circ \Delta_{\mathcal{C}} = \mathcal{C}.$$

Given an A -coring \mathcal{C} a *right \mathcal{C} -comodule* is a right A -module M together with a right A -module map $\rho^M : M \rightarrow M \otimes_A \mathcal{C}$, called a *coaction* such that

$$(\rho^M \otimes_A \mathcal{C}) \circ \rho^M = (M \otimes_A \Delta_{\mathcal{C}}) \circ \rho^M, \quad (M \otimes_A \epsilon_{\mathcal{C}}) \circ \rho^M = M.$$

A map between right \mathcal{C} -comodules is a right A -module map respecting the coactions, i.e., $f : M \rightarrow N$ satisfies $\rho^N \circ f = (f \otimes_A \mathcal{C}) \circ \rho^M$. The category of right \mathcal{C} -comodules is denoted by $\mathbf{M}^{\mathcal{C}}$, and their morphisms by $\text{Hom}^{\mathcal{C}}(-, -)$.

Example 2.1. ([27, 1.2 Example]) Let $B \rightarrow A$ be a ring extension. Then $\mathcal{C} = A \otimes_B A$ is an A -coring with the coproduct $\Delta_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C} \cong A \otimes_B A \otimes_B A$, $a \otimes_B a' \mapsto a \otimes_B 1 \otimes_B a'$ and the counit $\epsilon_{\mathcal{C}}(a \otimes_B a') = aa'$. \mathcal{C} is called the *canonical A -coring* associated to the extension $B \rightarrow A$.

Interesting examples of corings come from entwining structures. Recall from [7] that an *entwining structure* (over k) is a triple $(A, C)_{\psi}$ consisting of a k -algebra A , a k -coalgebra C and a k -module map $\psi : C \otimes A \rightarrow A \otimes C$ satisfying

$$\psi \circ (C \otimes \mu) = (\mu \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A), \quad (2.1)$$

$$(A \otimes \Delta) \circ \psi = (\psi \otimes C) \circ (C \otimes \psi) \circ (\Delta \otimes A), \quad (2.2)$$

$$\psi \circ (C \otimes 1) = 1 \otimes C, \quad (A \otimes \epsilon) \circ \psi = \epsilon \otimes A. \quad (2.3)$$

For $(A, C)_\psi$ we use the notation $\psi(c \otimes a) = a_\alpha \otimes c^\alpha$ (summation over a Greek index understood), for all $a \in A, c \in C$. Various examples of entwining structures can be found in [5]. Given an entwining structure $(A, C)_\psi$, an (entwined) $(A, C)_\psi$ -module is a right A -module, right C -comodule M such that

$$\rho^M \circ \rho_M = (\rho_M \otimes C) \circ (M \otimes \psi) \circ (\rho^M \otimes A), \quad (2.4)$$

where ρ_M is the action of A on M . Explicitly: $\rho^M(m \cdot a) = m_{(0)} \cdot a_\alpha \otimes m_{(1)}^\alpha$, $\forall a \in A, m \in M$. $A \otimes C$ is an example of entwined module with the coaction $A \otimes \Delta$ and the action $(a' \otimes c) \cdot a = a' \psi(c \otimes a)$, for all $a, a' \in A, c \in C$. The unifying Doi-Koppinen modules [14] [19] are another example of entwined modules (cf. [5, Example 3.1(3)]).

A morphism of $(A, C)_\psi$ -modules is a right A -module map which is also a right C -comodule map. The category of $(A, C)_\psi$ -modules is denoted by $\mathbf{M}_A^C(\psi)$.

The present paper is motivated by the following observation ascribed to M. Takeuchi [28]

Proposition 2.2. *For an entwining structure $(A, C)_\psi$, view $A \otimes C$ as an (A, A) -bimodule with the left action $a \cdot (a' \otimes c) = aa' \otimes c$ and the right action $(a' \otimes c) \cdot a = a' \psi(c \otimes a)$, for all $a, a' \in A, c \in C$. Then $\mathcal{C} = A \otimes C$ is an A -coring with the coproduct $\Delta_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C} \cong A \otimes C \otimes C$, $\Delta_{\mathcal{C}} = A \otimes \Delta$, and the counit $\epsilon_{\mathcal{C}} = A \otimes \epsilon$.*

Conversely if $A \otimes C$ is an A -coring with the coproduct, counit and left A -module structure above, then $(A, C)_\psi$ is an entwining structure, where $\psi : c \otimes a \mapsto (1 \otimes c) \cdot a$.

Under this bijective correspondence $\mathbf{M}^{\mathcal{C}} = \mathbf{M}_A^C(\psi)$.

Proof. One can easily check that given an entwining structure $(A, C)_\psi$ one has the coring $A \otimes C$ as described in the proposition. Conversely, let $A \otimes C$ be an A -coring with structure maps given in the proposition. The properties of the right A -action imply equation (2.1) and the first of equations (2.3) required for the entwining map ψ . The remaining two conditions follow from the facts that $A \otimes \Delta$ and $A \otimes \epsilon$ are right A -module maps respectively. \square

Recall from [6] that given a coalgebra C , an extension of algebras $B \subset A$ is C -Galois if A is a right C -comodule with coaction ρ^A , $B = A^{\text{co}C} = \{b \in A \mid \forall a \in A, \rho^A(ba) = b\rho^A(a)\}$, and the canonical left A -module, right C -comodule map $\text{can} : A \otimes_B A \rightarrow A \otimes C$, $a \otimes_B a' \mapsto a\rho^A(a')$ is bijective. In this case, since $A \otimes_B A$ is an A -coring by Example 2.1, the canonical map can induces an A -coring structure on $A \otimes C$. The corresponding entwining map ψ computed from Proposition 2.2 comes out as $\psi(c \otimes a) = \text{can}(\text{can}^{-1}(1 \otimes c)a)$, for all $a \in A$ and $c \in C$ and thus coincides with the canonical entwining structure associated to a C -Galois extension $B \subset A$ constructed in [6, Theorem 2.7].

In the context of C -Galois extensions one can also mention the following example of a coring inspired by [25, 3.5 Theorem]. Given a C -Galois extension $B \subset A$ define a (B, B) -bimodule

$$\mathcal{C} = \left\{ \sum_i a^i \otimes \bar{a}^i \in A \otimes A \mid \sum_i a^i_{(0)} \otimes \text{can}^{-1}(1 \otimes a^i_{(1)}) \bar{a}^i = \sum_i a^i \otimes \bar{a}^i \otimes_B 1 \right\}.$$

If A is faithfully flat as both k and B -bimodule then \mathcal{C} is a B -coring with the coproduct and counit

$$\Delta_{\mathcal{C}}\left(\sum_i a^i \otimes \bar{a}^i\right) = \sum_i a^i_{(0)} \otimes \text{can}^{-1}(1 \otimes a^i_{(1)}) \otimes \bar{a}^i, \quad \epsilon_{\mathcal{C}}\left(\sum_i a^i \otimes \bar{a}^i\right) = \sum_i a^i \bar{a}^i.$$

The fact that \mathcal{C} is a coring can be verified by direct computation which uses properties of the canonical map can .

A generalisation of the notion of an entwining structure is possible by replacing equations (2.3) by weaker conditions, for all $a \in A$, $c \in C$,

$$a_{\alpha} \epsilon(c^{\alpha}) = 1_{\alpha} a \epsilon(c^{\alpha}), \quad 1_{\alpha} \epsilon(c_{(1)}^{\alpha}) \otimes c_{(2)} = 1_{\alpha} \otimes c^{\alpha}, \quad (2.5)$$

where $\psi(c \otimes a) = a_{\alpha} \otimes c^{\alpha}$. Such a weakened entwining structure is termed a *weak entwining structure* [9] as it includes examples coming from weak Hopf algebras [1][2], and is denoted by (A, C, ψ) . Given a weak entwining structure (A, C, ψ) one can define the category of weak entwined modules as right A -modules, right C -comodules such that eq. (2.4) holds. We denote this category by $\mathbf{M}_A^C(\psi)$ too.

Proposition 2.3. *Let (A, C, ψ) be a weak entwining structure. Let $p : A \otimes C \rightarrow A \otimes C$, $p = (\mu \otimes C) \circ (A \otimes \psi) \circ (A \otimes C \otimes 1)$, and $\mathcal{C} = \text{Imp} = \{\sum_i a_i 1_{\alpha} \otimes c_i^{\alpha} \mid \sum_i a_i \otimes c_i \in A \otimes C\}$. Then $p \circ p = p$ and*

(1) \mathcal{C} is an (A, A) -bimodule with the left action $a' \cdot (a1_\alpha \otimes c^\alpha) = a'a1_\alpha \otimes c^\alpha$ and the right action $(a'1_\alpha \otimes c^\alpha) \cdot a = a'1_\alpha a_\beta \otimes c^{\alpha\beta} = a'a_\alpha \otimes c^\alpha$ (this last equality follows from equation (2.1)).

(2) \mathcal{C} is an A -coring with coproduct $\Delta_{\mathcal{C}} = (A \otimes \Delta) |_{\mathcal{C}}$ and counit $\epsilon_{\mathcal{C}} = (A \otimes \epsilon) |_{\mathcal{C}}$.

(3) $\mathbf{M}_A^{\mathcal{C}}(\psi) = \mathbf{M}^{\mathcal{C}}$

Proof. For any $a \otimes c \in A \otimes \mathcal{C}$, using the definition of p and equation (2.1) we have

$$p \circ p(a \otimes c) = p(a1_\alpha \otimes c^\alpha) = a1_\alpha 1_\beta \otimes c^{\alpha\beta} = a1_\alpha \otimes c^\alpha = p(a \otimes c).$$

Therefore p is a projection as claimed.

(1) Clearly \mathcal{C} is a left A -module. The fact that it is a right A -module can be checked directly using equations (2.1) and (2.5).

(2) First we need to show that $\text{Im} \Delta_{\mathcal{C}} \subset \mathcal{C} \otimes_A \mathcal{C}$. Since $\Delta_{\mathcal{C}}$ is k -linear and, evidently, left A -linear, it suffices to take an element of \mathcal{C} of the form $e = 1_\alpha \otimes c^\alpha$ and compute

$$\begin{aligned} \Delta_{\mathcal{C}}(e) &= 1_\alpha \otimes c^\alpha_{(1)} \otimes c^\alpha_{(2)} = 1_{\alpha\beta} \otimes c_{(1)}^\beta \otimes c_{(2)}^\alpha = 1_\gamma 1_{\alpha\beta} \otimes c_{(1)}^{\gamma\beta} \otimes c_{(2)}^\alpha \\ &= (1_\gamma \otimes c_{(1)}^\gamma) \cdot 1_\alpha \otimes c_{(2)}^\alpha = (1_\gamma \otimes c_{(1)}^\gamma) \otimes_A (1_\alpha \otimes c_{(2)}^\alpha) \in \mathcal{C} \otimes_A \mathcal{C}. \end{aligned}$$

We used equations (2.2), (2.1) and the fact that 1 is a unit in A to derive second and third equalities, and then definition of the right action of A on \mathcal{C} to obtain the fourth one.

To prove that $\Delta_{\mathcal{C}}$ is a right A -module map it is enough to consider any $e \in \mathcal{C}$ of the above form, use the same equations as for the preceding calculation, and compute for all $a \in A$,

$$\begin{aligned} \Delta_{\mathcal{C}}(e \cdot a) &= a_\alpha \otimes c^\alpha_{(1)} \otimes c^\alpha_{(2)} = a_{\alpha\beta} \otimes c_{(1)}^\beta \otimes c_{(2)}^\alpha = (1_\gamma \otimes c_{(1)}^\gamma) \otimes_A (a_\alpha \otimes c_{(2)}^\alpha) \\ &= (1_\gamma \otimes c_{(1)}^\gamma) \otimes_A (1_\alpha \otimes c_{(2)}^\alpha) \cdot a = \Delta_{\mathcal{C}}(e) \cdot a. \end{aligned}$$

The coassociativity of $\Delta_{\mathcal{C}}$ follows from the coassociativity of the coproduct Δ . Now, it is clear from the definition that $\epsilon_{\mathcal{C}}$ is a left A -module map. To prove that it is a right A -module map as well take $e \in \mathcal{C}$ as above and compute for all $a \in A$,

$$\epsilon_{\mathcal{C}}(e \cdot a) = a_\alpha \epsilon_{\mathcal{C}}(c^\alpha) = 1_\alpha a \epsilon_{\mathcal{C}}(c^\alpha) = \epsilon_{\mathcal{C}}(e) a,$$

where the first of equations (2.5) was used to obtain the second equality. Again, directly from the construction of ϵ_C it follows that $(C \otimes_A \epsilon_C) \circ \Delta_C = C$. On the other hand using the definitions of Δ_C and ϵ_C , the second of equations (2.5) and equation (2.1) we obtain for any $e \in C$ of the above form

$$(\epsilon_C \otimes_A C) \circ \Delta_C(e) = \epsilon_C(1_\alpha \otimes c_{(1)}^\alpha) \cdot (1_\beta \otimes c_{(2)}^\beta) = 1_\alpha 1_\beta \epsilon(c_{(1)}^\alpha) \otimes c_{(2)}^\beta = 1_\alpha 1_\beta \otimes c^{\alpha\beta} = e.$$

This completes the proof that C is an A -coring.

(3) If $M \in \mathbf{M}_A^C(\psi)$ with the right C -coaction $m \mapsto m_{(0)} \otimes m_{(1)}$, then $M \in \mathbf{M}^C$ with the coaction $\rho^M(m) = m_{(0)} \otimes_A (1_\alpha \otimes m_{(1)}^\alpha) = m_{(0)} \cdot 1_\alpha \otimes m_{(1)}^\alpha = m_{(0)} \otimes m_{(1)}$ since $M \in \mathbf{M}_A^C(\psi)$. Conversely, if $M \in \mathbf{M}^C$ then the coaction $\rho^M : M \rightarrow M \otimes_A C \subset M \otimes C$ provides M also with the weak entwined module structure. \square

An example of a weak entwining structure can be obtained by the following construction.

Example 2.4. Let C be a coalgebra, A an algebra and a right C -comodule with the coaction ρ^A . Let $B = A^{coC} = \{b \in A \mid \forall a \in A, \rho^A(ba) = b\rho^A(a)\}$, and let $can : A \otimes_B A \rightarrow A \otimes C$, $a \otimes_B a' \mapsto a\rho^A(a')$. View $A \otimes_B A$ as a left A -module via $\mu \otimes_B A$ and a right C -comodule via $A \otimes_B \rho^A$. View $A \otimes C$ as a left A -module via $\mu \otimes C$ and as a right C -comodule via $A \otimes \Delta$. Now suppose that can is a split monomorphism in the category of left A -modules and right C -comodules, i.e., there exists left A -module, right C -comodule map $\sigma : A \otimes C \rightarrow A \otimes_B A$ such that $\sigma \circ can = A \otimes_B A$. Let $\tau : C \rightarrow A \otimes_B A$, $c \mapsto \sigma(1 \otimes c)$. Define

$$\psi : C \otimes A \rightarrow A \otimes C, \quad \psi = can \circ (A \otimes_B \mu) \circ (\tau \otimes A).$$

Then (A, C, ψ) is a weak entwining structure. The extension of algebras $B \subset A$ is called a *weak C -Galois extension*.

Example 2.4 can be proven by a direct verification of the axioms in a way similar to the proof of [6, Theorem 2.7]. Similarly one can prove that in this case A is a weak entwined (A, C, ψ) -module via ρ^A and multiplication in A , and that (A, C, ψ) is unique with this respect.

The fact that given a (weak) entwining structure $(A, C)_\psi$, there is an associated A -coring C explains various properties of entwining structures. For example, the

existence of the ψ -twisted convolution product on $\text{Hom}(C, A)$ given by $f *_{\psi} g(c) = f(c_{(2)})_{\alpha} g(c_{(1)}^{\alpha})$ is the consequence of the product on ${}_A\text{Hom}(\mathcal{C}, A)$ defined in [27, 3.2 Proposition (a)]. We deal with this product in Section 4. In the remaining part of this paper we show that some properties of entwined modules can be derived from properties of comodules of a coring.

3. SEPARABLE FUNCTORS FOR A CORING AND A MASCHKE-TYPE THEOREM

In this section, for an A -coring \mathcal{C} we study when two functors, the forgetful functor $\mathbf{M}^{\mathcal{C}} \rightarrow \mathbf{M}_A$ and the induction functor $- \otimes_A \mathcal{C} : \mathbf{M}_A \rightarrow \mathbf{M}^{\mathcal{C}}$, $M \mapsto M \otimes_A \mathcal{C}$ are separable. The right \mathcal{C} -comodule structure of $M \otimes_A \mathcal{C}$ is given by $M \otimes_A \Delta_{\mathcal{C}}$. Recall from [22] that a covariant functor $F : \mathbf{C} \rightarrow \mathbf{D}$ is *separable* if the natural transformation $\text{Hom}_{\mathbf{C}}(-, -) \rightarrow \text{Hom}_{\mathbf{D}}(F(-), F(-))$ splits. We start by observing the following fact [15, Proposition 3.1].

Lemma 3.1. *The functor $- \otimes_A \mathcal{C}$ is the right adjoint of the forgetful functor $F : \mathbf{M}^{\mathcal{C}} \rightarrow \mathbf{M}_A$.*

Proof. For any $M \in \mathbf{M}_A$ define, $\Psi_M : M \otimes_A \mathcal{C} \rightarrow M$, $\Psi_M = M \otimes_A \epsilon_{\mathcal{C}}$, while for any $N \in \mathbf{M}^{\mathcal{C}}$ define $\Phi_N : N \rightarrow N \otimes_A \mathcal{C}$, $\Phi_N = \rho^N$. Clearly both Ψ_M and Φ_N are natural in M and N respectively. Furthermore, because ρ^N is a coaction $\Psi_N \circ \Phi_N = N$, for all $N \in \mathbf{M}^{\mathcal{C}}$. It remains to be shown that for all right A -modules M , $(\Psi_M \otimes_A \mathcal{C}) \circ \Phi_{M \otimes_A \mathcal{C}} = M \otimes_A \mathcal{C}$. Take any $m \in M$ and $c \in \mathcal{C}$ and compute

$$(\Psi_M \otimes_A \mathcal{C}) \circ \Phi_{M \otimes_A \mathcal{C}}(m \otimes_A c) = m \otimes_A \epsilon_{\mathcal{C}}(c_{(1)}) \cdot c_{(2)} = m \otimes_A c.$$

Thus Ψ and Φ are the required adjunctions. \square

In particular if \mathcal{C} is taken as in Proposition 2.3, Lemma 3.1 implies [9, Proposition 2.2]. We now study when the functors in Lemma 3.1 are separable. Since we are dealing with a pair of adjoint functors, the following characterisation of separable functors, obtained in [24] [13], is of great importance

Theorem 3.2. *Let $G : \mathbf{D} \rightarrow \mathbf{C}$ be the right adjoint of $F : \mathbf{C} \rightarrow \mathbf{D}$ with adjunctions $\Phi : \mathbf{C} \rightarrow GF$ and $\Psi : FG \rightarrow \mathbf{D}$. Then*

(1) F is separable if and only if Φ splits, i.e., for all objects $C \in \mathbf{C}$ there exists a morphism $\nu_C \in \text{Mor}_{\mathbf{C}}(GF(C), C)$ such that $\nu_C \circ \Phi_C = C$ and for all $f \in \text{Mor}_{\mathbf{C}}(C, \tilde{C})$, $\nu_{\tilde{C}} \circ GF(f) = f \circ \nu_C$.

(2) G is separable if and only if Ψ cosplits, i.e., for all objects $D \in \mathbf{D}$ there exists a morphism $\nu_D \in \text{Mor}_{\mathbf{D}}(D, FG(D))$ such that $\Psi_D \circ \nu_D = D$ and for all $f \in \text{Mor}_{\mathbf{D}}(D, \tilde{D})$, $\nu_{\tilde{D}} \circ f = FG(f) \circ \nu_D$.

First we analyse when $- \otimes_A \mathcal{C}$ is a separable functor.

Theorem 3.3. *Let \mathcal{C} be an A -coring. Then the functor $- \otimes_A \mathcal{C}$ is separable if and only if there exists an invariant $e \in \mathcal{C}^A$ such that $\epsilon_{\mathcal{C}}(e) = 1$.*

Proof. “ \Rightarrow ”. Suppose that $- \otimes_A \mathcal{C}$ is separable, and let ν be split by Ψ , the latter defined in Lemma 3.1. Since A is a right A -module we can take $\nu_A \in \text{Hom}_A(A, A \otimes_A \mathcal{C}) \cong \text{Hom}_A(A, \mathcal{C})$ and define $e = \nu_A(1) \in \mathcal{C}$. Since ν_A is split by $\Psi_A = A \otimes_A \epsilon_{\mathcal{C}} = \epsilon_{\mathcal{C}}$, we have $1 = \Psi_A \circ \nu_A(1) = \epsilon_{\mathcal{C}} \circ \nu_A(1) = \epsilon_{\mathcal{C}}(e)$. Now for any $a \in A$ define the morphism $f_a \in \text{Hom}_A(A, A)$ by $f_a(a') = aa'$. Since ν_A is natural we have:

$$\nu_A(aa') = \nu_A \circ f_a(a') = (f_a \otimes_A \mathcal{C}) \circ \nu_A(a') = a \cdot \nu_A(a').$$

This implies, in particular, $\nu_A(a) = a \cdot \nu_A(1) = a \cdot e$. On the other hand, ν_A is a right A -module morphism, thus we have, $a \cdot e = \nu_A(a) = \nu_A(1) \cdot a = e \cdot a$, so that e is A -central as required.

“ \Leftarrow ”. Suppose there exists $e \in \mathcal{C}^A$ such that $\epsilon_{\mathcal{C}}(e) = 1$. For any $M \in \mathbf{M}_A$ define $\nu_M : M \rightarrow M \otimes_A \mathcal{C}$ by $\nu_M(m) = m \otimes_A e$. Since e is A -central we have for all $a \in A$, $m \in M$:

$$\nu_M(m \cdot a) = m \cdot a \otimes_A e = m \otimes_A a \cdot e = m \otimes_A e \cdot a = \nu_M(m) \cdot a,$$

hence ν_M is a right A -module map. Furthermore $\Psi_M \circ \nu_M(m) = m \cdot \epsilon_{\mathcal{C}}(e) = m$, i.e., ν_M is split by Ψ_M . Finally for any $f \in \text{Hom}_A(M, N)$ we have $\nu_N \circ f(m) = f(m) \otimes_A e = (f \otimes_A \mathcal{C}) \circ \nu_M(m)$, which means that ν_M is natural in M . \square

One of the motivations for the introduction of separable functors was that in the case of a ring extension $B \rightarrow A$, the restriction of scalars functor is separable if

and only if the extension $B \rightarrow A$ is separable [22]. Theorem 3.2 provides one with different characterisation of separable extensions.

Corollary 3.4. *Let $B \rightarrow A$ be a ring extension and let \mathcal{C} be the canonical A -coring in Example 2.1. The extension $B \rightarrow A$ is separable if and only if the functor $- \otimes_A \mathcal{C}$ is separable.*

Proof. Recall the a ring extension $B \rightarrow A$ is separable iff there exists an invariant $e = \sum_i a_i \otimes_B a'_i \in (A \otimes_B A)^A$ such that $\sum_i a_i a'_i = 1$ (cf. [16, Definition 2]). Since in the canonical coring $A \otimes_B A$ the counit is given by the multiplication projected to $A \otimes_B A$ the separability of $B \rightarrow A$ is equivalent to the separability of $- \otimes_A \mathcal{C}$ by the preceding theorem. \square

In the case of a coring $\mathcal{C} = A \otimes C$ corresponding to an entwining structure $(A, C)_\psi$ via Proposition 2.2, a normalised A -central element $e \in \mathcal{C}$ is simply an integral in $(A, C)_\psi$ in the sense of [3, Definition 3.1]. Thus Theorem 3.3 implies [3, Theorem 3.3] (which itself is a generalisation of [11, Theorem 2.14] formulated for Doi-Koppinen modules) while Corollary 3.4 implies [3, Proposition 3.5].

Theorem 3.5. *For an A -coring \mathcal{C} , the forgetful functor $F : \mathbf{M}^{\mathcal{C}} \rightarrow \mathbf{M}_A$ is separable if and only if there exists $\gamma \in {}_A \text{Hom}_A(\mathcal{C} \otimes_A \mathcal{C}, A)$ such that $\gamma \circ \Delta_{\mathcal{C}} = \epsilon_{\mathcal{C}}$ and*

$$c_{(1)} \cdot \gamma(c_{(2)} \otimes_A c') = \gamma(c \otimes_A c'_{(1)}) \cdot c'_{(2)}, \quad \forall c, c' \in \mathcal{C}. \quad (3.6)$$

Proof. “ \Rightarrow ”. Suppose F is separable. Let Φ be the adjunction defined in Lemma 3.1 and let ν be the splitting of Φ . Since \mathcal{C} is a right \mathcal{C} -comodule via $\Delta_{\mathcal{C}}$ there is a corresponding $\nu_{\mathcal{C}}$ and we can define $\gamma = \epsilon_{\mathcal{C}} \circ \nu_{\mathcal{C}} : \mathcal{C} \otimes_A \mathcal{C} \rightarrow A$. The map γ is a right A -module morphism as a composition of two such morphisms. Next for all $a \in A$ consider $f_a \in \text{Hom}^{\mathcal{C}}(\mathcal{C}, \mathcal{C})$ given by $f_a(c) = a \cdot c$. The naturality of ν implies for all $c, c' \in \mathcal{C}$, $\nu_{\mathcal{C}}(f_a(c) \otimes_A c') = f_a \circ \nu_{\mathcal{C}}(c \otimes_A c')$, i.e., $\nu_{\mathcal{C}}(a \cdot c \otimes_A c') = a \cdot \nu_{\mathcal{C}}(c \otimes_A c')$. Therefore $\nu_{\mathcal{C}}$ is a left A -module map, and, consequently, γ is a left A -module morphism as a composition of two such morphisms. Since $\nu_{\mathcal{C}}$ splits $\Phi_{\mathcal{C}} = \Delta_{\mathcal{C}}$ we have

$$\gamma \circ \Delta_{\mathcal{C}} = \epsilon_{\mathcal{C}} \circ \nu_{\mathcal{C}} \circ \Delta_{\mathcal{C}} = \epsilon_{\mathcal{C}}.$$

Now, for all $c \in \mathcal{C}$ consider the morphism $\ell_c \in \text{Hom}^{\mathcal{C}}(\mathcal{C}, \mathcal{C} \otimes_A \mathcal{C})$, $\ell_c(c') = c \otimes_A c'$, and also the morphism $\Delta_{\mathcal{C}} \in \text{Hom}^{\mathcal{C}}(\mathcal{C}, \mathcal{C} \otimes_A \mathcal{C})$. By the naturality of $\nu_{\mathcal{C}}$ we have

$$\ell_c \circ \nu_{\mathcal{C}} = \nu_{\mathcal{C} \otimes_A \mathcal{C}} \circ (\ell_c \otimes_A \mathcal{C}), \quad \Delta_{\mathcal{C}} \circ \nu_{\mathcal{C}} = \nu_{\mathcal{C} \otimes_A \mathcal{C}} \circ (\Delta_{\mathcal{C}} \otimes_A \mathcal{C}). \quad (*)$$

From the first of equations $(*)$ we have for all $c, c', c'' \in \mathcal{C}$, $c \otimes_A \nu_{\mathcal{C}}(c' \otimes_A c'') = \nu_{\mathcal{C} \otimes_A \mathcal{C}}(c \otimes_A c' \otimes_A c'')$. Combining this expression with the second of equations $(*)$ we obtain

$$\Delta_{\mathcal{C}}(\nu_{\mathcal{C}}(c \otimes_A c')) = \nu_{\mathcal{C} \otimes_A \mathcal{C}}(c_{(1)} \otimes_A c_{(2)} \otimes_A c') = c_{(1)} \otimes_A \nu_{\mathcal{C}}(c_{(2)} \otimes_A c').$$

Applying $\mathcal{C} \otimes_A \epsilon_{\mathcal{C}}$ to this equality one obtains

$$\nu_{\mathcal{C}}(c \otimes_A c') = c_{(1)} \cdot \gamma(c_{(2)} \otimes_A c').$$

On the other hand, $\nu_{\mathcal{C}}$ is a right \mathcal{C} -comodule map, so that $\Delta_{\mathcal{C}}(\nu_{\mathcal{C}}(c \otimes_A c')) = \nu_{\mathcal{C}}(c \otimes_A c'_{(1)}) \otimes_A c'_{(2)}$. Applying $\epsilon_{\mathcal{C}} \otimes_A \mathcal{C}$ to this formula we obtain

$$\nu_{\mathcal{C}}(c \otimes_A c') = \gamma(c \otimes_A c'_{(1)}) \cdot c'_{(2)}.$$

Thus $\nu_{\mathcal{C}}$ can be expressed in terms of γ in two different ways. Comparison gives eq. (3.6) as required.

“ \Leftarrow ”. Suppose there exists γ as in the theorem. Then for all $M \in \mathbf{M}^{\mathcal{C}}$ define an additive map $\nu_M : M \otimes_A \mathcal{C} \rightarrow M$, $m \otimes_A c \mapsto m_{(0)} \cdot \gamma(m_{(1)} \otimes_A c)$. Clearly ν_M is a right A -module map since γ is such a map. Furthermore, for any $m \in M$, $c \in \mathcal{C}$ we have

$$\begin{aligned} \nu_M(m \otimes_A c_{(1)}) \otimes_A c_{(2)} &= m_{(0)} \cdot \gamma(m_{(1)} \otimes_A c_{(1)}) \otimes_A c_{(2)} \\ &= m_{(0)} \otimes_A \gamma(m_{(1)} \otimes_A c_{(1)}) \cdot c_{(2)} \\ &= m_{(0)} \otimes_A m_{(1)} \cdot \gamma(m_{(2)} \otimes_A c) \\ &= \nu_M(m \otimes_A c)_{(0)} \otimes_A \nu_M(m \otimes_A c)_{(1)}, \end{aligned}$$

where we used eq. (3.6). This means that ν_M is a morphism in $\mathbf{M}^{\mathcal{C}}$. Furthermore, take any $f \in \text{Hom}^{\mathcal{C}}(M, N)$. Then for all $m \in M$, $c \in \mathcal{C}$ we have

$$\nu_M(f(m) \otimes_A c) = f(m)_{(0)} \cdot \gamma(f(m)_{(1)} \otimes_A c) = f(m_{(0)}) \cdot \gamma(m_{(1)} \otimes_A c) = f \circ \nu_M(m \otimes_A c),$$

where we used the fact that f is a right \mathcal{C} -comodule map. Finally we take any $m \in M$ and compute

$$\nu_M \circ \Phi_M(m) = \nu_M(m_{(0)} \otimes_A m_{(1)}) = m_{(0)} \cdot \gamma(m_{(1)} \otimes_A m_{(2)}) = m_{(0)} \cdot \epsilon_{\mathcal{C}}(m_{(1)}) = m.$$

This shows that ν is the required splitting of Φ . \square

Recall from [15] that an A -coring \mathcal{C} is said to be *coseparable* if there exists a $(\mathcal{C}, \mathcal{C})$ -bicomodule splitting of the coproduct. Explicitly one requires an (A, A) -bimodule map $\pi : \mathcal{C} \otimes_A \mathcal{C} \rightarrow \mathcal{C}$ such that

$$(\mathcal{C} \otimes_A \pi) \circ (\Delta_{\mathcal{C}} \otimes_A \mathcal{C}) = \Delta_{\mathcal{C}} \circ \pi = (\pi \otimes_A \mathcal{C}) \circ (\mathcal{C} \otimes_A \Delta_{\mathcal{C}}), \quad \pi \circ \Delta_{\mathcal{C}} = \mathcal{C}. \quad (3.7)$$

Theorem 3.5 implies the following characterisation of coseparable corings which supplements [15, Theorem 3.10].

Corollary 3.6. *An A -coring \mathcal{C} is coseparable if and only if the forgetful functor $F : \mathbf{M}^{\mathcal{C}} \rightarrow \mathbf{M}_A$ is separable.*

Proof. Given a map γ as in Theorem 3.5 one defines $\pi : \mathcal{C} \otimes_A \mathcal{C} \rightarrow \mathcal{C}$, $c \otimes_A c' \mapsto c_{(1)} \cdot \gamma(c_{(2)} \otimes_A c')$. By the “ \Leftarrow ” part of the proof of Theorem 3.5, $\pi = \nu_{\mathcal{C}}$ and thus it is a right \mathcal{C} -comodule splitting of $\Delta_{\mathcal{C}}$. Using the fact that $\Delta_{\mathcal{C}}$ is an (A, A) -bimodule map one easily verifies that π is a left \mathcal{C} -comodule map. Conversely, given π define $\gamma = \epsilon_{\mathcal{C}} \circ \pi$. Since π splits $\Delta_{\mathcal{C}}$, $\gamma \circ \Delta_{\mathcal{C}} = \epsilon_{\mathcal{C}}$. Applying $\mathcal{C} \otimes_A \epsilon_{\mathcal{C}}$ to the first equality in (3.7) and $\epsilon_{\mathcal{C}} \otimes_A \mathcal{C}$ to the second equality in (3.7) one deduces eq. (3.6). By Theorem 3.5, the forgetful functor is separable as required. \square

Corollary 3.7. *Let $B \rightarrow A$ be a ring extension such that A is faithfully flat as either left or right B -module, and let \mathcal{C} be the canonical A -coring in Example 2.1. The extension $B \rightarrow A$ is split if and only if the functor $F : \mathbf{M}^{\mathcal{C}} \rightarrow \mathbf{M}_A$ is separable.*

Proof. Recall that an extension $B \rightarrow A$ is split iff there exists a (B, B) -bimodule map $E : A \rightarrow B$ such that $E(1) = 1$ (cf. [23]). In the case of the canonical A -coring $A \otimes_B A$ the conditions required for the map $\gamma \in {}_A\text{Hom}_A(A \otimes_B A \otimes_B A, A)$ read $\gamma(a \otimes_B 1 \otimes_B a') = aa'$ and $a \otimes_B \gamma(1 \otimes_B a' \otimes_B a'') = \gamma(a \otimes_B a' \otimes_B 1) \otimes_B a''$ for all $a, a', a'' \in A$. Since ${}_A\text{Hom}_A(A \otimes_B A \otimes_B A, A) \cong {}_B\text{Hom}_B(A, A)$ the maps γ are in

one-to-one correspondence with the maps $E \in {}_B\text{Hom}_B(A, A)$ via $\gamma(a \otimes_B a' \otimes_B a'') = aE(a')a''$. The first of the above conditions for γ is equivalent to the normalisation of E , $E(1) = 1$, while the second condition gives for all $a \in A$, $1 \otimes_B E(a) = E(a) \otimes_B 1$. By the faithfully flat descent the latter is equivalent to $E(a) \in B$. \square

In the case of the coring $\mathcal{C} = A \otimes C$ corresponding to an entwining structure $(A, C)_\psi$ via Proposition 2.2, a map γ in Theorem 3.5 corresponds to a normalised integral map in $(A, C)_\psi$ in the sense of [3, Definition 4.1]. Thus Theorem 3.5 implies [3, Theorem 4.2], the latter being a generalisation of [11, Theorem 2.3] formulated for Doi-Koppinen modules.

As explained in [11] (cf. [22, Proposition 1.2]) the separability of the forgetful functor implies various Maschke-type theorems. Thus, we have the following generalisation of [3, Corollary 4.4]

Corollary 3.8. (Maschke-type theorem for a coring) *Let \mathcal{C} be an A -coring with a map γ satisfying hypothesis of Theorem 3.5. Then every right \mathcal{C} -comodule which is semisimple (resp. projective, injective) as a right A -module is semisimple (resp. projective, injective) \mathcal{C} -comodule.*

4. FROBENIUS PROPERTIES OF A CORING

Let \mathcal{C} be an A -coring. As explained in [27, 3.2 Proposition (a)], $R = {}_A\text{Hom}(\mathcal{C}, A)$ is a ring with unit $\epsilon_{\mathcal{C}}$ and product $(rr')(c) = r'(c_{(1)} \cdot r(c_{(2)}))$, for all $r, r' \in R$ and $c \in \mathcal{C}$. R is a left A -module via $(a \cdot r)(c) = r(c \cdot a)$, for all $a \in A, c \in \mathcal{C}, r \in R$. Furthermore the map $\iota : A \rightarrow R$ given by $\iota(a)(c) = \epsilon_{\mathcal{C}}(c)a$ is a ring map. In this section we study when $A \rightarrow R$ is a Frobenius extension.

Recall from [18] and [21] that a ring extension $A \rightarrow R$ is called a *Frobenius extension* (of the first kind) iff R is a finitely generated projective right A -module and $R \cong \text{Hom}_A(R, A)$ as (A, R) -bimodules. The (A, R) -bimodule structure of $\text{Hom}_A(R, A)$ is given by $(a \cdot f \cdot r)(r') = af(rr')$, for all $a \in A, r, r' \in R$ and $f \in \text{Hom}_A(R, A)$. Frobenius extensions are closely related with a certain type of corings, namely, $A \rightarrow R$ is a Frobenius extension if and only if R is an A -coring

such that the coproduct is an (R, R) -bimodule map (cf. [17, Proposition 4.3], [10, Remark 2.5]). The main result of this section is contained in the following

Theorem 4.1. *Let \mathcal{C} be an A -coring and let $R = {}_A\text{Hom}(\mathcal{C}, A)$. If \mathcal{C} is a projective left A -module then the following are equivalent:*

- (1) *The forgetful functor $F : \mathbf{M}^{\mathcal{C}} \rightarrow \mathbf{M}_A$ is a Frobenius functor.*
- (2) *\mathcal{C} is a finitely generated left A -module and the ring extension $A \rightarrow R$ is Frobenius.*
- (3) *\mathcal{C} is a finitely generated left A -module and $\mathcal{C} \cong R$ as (A, R) -bimodules, where \mathcal{C} is a right R -module via $c \cdot r = c_{(1)} \cdot r(c_{(2)})$, $\forall c \in \mathcal{C}, r \in R$.*
- (4) *\mathcal{C} is a finitely generated left A -module and there exists $e \in \mathcal{C}^A$ such that the map $\phi : R \rightarrow \mathcal{C}$, $r \mapsto e_{(1)} \cdot r(e_{(2)})$ is bijective.*

Recall from [12] that a functor is said to be *Frobenius* in case it has the same right and left adjoint. Since by Lemma 3.1 the functor $- \otimes_A \mathcal{C} : \mathbf{M}_A \rightarrow \mathbf{M}^{\mathcal{C}}$ is the right adjoint of the forgetful functor F , Theorem 4.1 (1) is equivalent to the statement that $- \otimes_A \mathcal{C}$ is the left adjoint of F . Proof of Theorem 4.1 is based on two lemmas

Lemma 4.2. *If $- \otimes_A \mathcal{C} : \mathbf{M}_A \rightarrow \mathbf{M}^{\mathcal{C}}$ is the left adjoint of the forgetful functor $F : \mathbf{M}^{\mathcal{C}} \rightarrow \mathbf{M}_A$, then \mathcal{C} is a finitely generated left A -module.*

Proof. The proof of this lemma is based on the proofs of [12, Lemma 2.3, Theorem 2.4 1) \Rightarrow 2)]. Let for all $M \in \mathbf{M}_A$, $N \in \mathbf{M}^{\mathcal{C}}$, $\eta_{M,N} : \text{Hom}^{\mathcal{C}}(M \otimes_A \mathcal{C}, N) \rightarrow \text{Hom}_A(M, N)$ be the natural isomorphism and let $e = \eta_{M,N}(\mathcal{C})(1)$. Since η is natural and $A \in \mathbf{M}_A$, $\mathcal{C} \in \mathbf{M}^{\mathcal{C}}$ we have for all $f \in \text{Hom}^{\mathcal{C}}(\mathcal{C}, \mathcal{C})$, $\eta_{A,\mathcal{C}} \circ \text{Hom}^{\mathcal{C}}(\mathcal{C}, f) = \text{Hom}_A(A, f) \circ \eta_{A,\mathcal{C}}$. Evaluating this equality at \mathcal{C} and the resulting equality at 1 we obtain $\eta_{A,\mathcal{C}}(f)(1) = f(\eta_{A,\mathcal{C}}(\mathcal{C})(1)) = f(e)$. Now, for any $c \in \mathcal{C}$, let $f_c \in \text{Hom}^{\mathcal{C}}(\mathcal{C}, \mathcal{C})$ be the unique morphism in $\mathbf{M}^{\mathcal{C}}$ such that $\eta_{A,\mathcal{C}}(f_c)(a) = c \cdot a$, $\forall a \in A$. Taking $a = 1$ and using above equality we obtain $c = \eta_{A,\mathcal{C}}(f_c)(1) = f_c(e)$. Let $\Delta_{\mathcal{C}}(e) = \sum_{i=1}^n e_i \otimes_A \bar{e}_i$. Since f_c is a right \mathcal{C} -comodule map we have $c_{(1)} \otimes_A c_{(2)} = \sum_{i=1}^n f_c(e_i) \otimes_A \bar{e}_i$. Applying $\epsilon_{\mathcal{C}} \otimes_A \mathcal{C}$ to this equality we obtain for all $c \in \mathcal{C}$, $c = \sum_{i=1}^n \epsilon_{\mathcal{C}}(f_c(e_i)) \cdot \bar{e}_i$, i.e., \mathcal{C} is a finitely generated left A -module. \square

In fact using the same techniques as in the proof of [12, Lemma 2.3] one can establish that with the notation and hypothesis of Lemma 4.2, $\eta_{M,N}(f)(m) = f(m \otimes_A e)$, for all $f \in \text{Hom}^{\mathcal{C}}(M \otimes_A \mathcal{C}, N)$, $m \in M$.

Lemma 4.3. *Let \mathcal{C} be an A -coring and let $R = {}_A\text{Hom}(\mathcal{C}, A)$. If \mathcal{C} is a finitely generated projective left A -module then the categories $\mathbf{M}^{\mathcal{C}}$ and \mathbf{M}_R are isomorphic with each other.*

Proof. Given $M \in \mathbf{M}^{\mathcal{C}}$ one can view it as a right R -module via $m \cdot r = m_{(0)} \cdot r(m_{(1)})$, for all $m \in M$, $r \in R$. Indeed, it is immediate that $m \cdot \epsilon_{\mathcal{C}} = m$. Furthermore, using that the right coaction of \mathcal{C} on M is a right A -module map we have for all $r, r' \in R$, $m \in M$,

$$(m \cdot r) \cdot r' = (m \cdot r)_{(0)} \cdot r'((m \cdot r)_{(1)}) = m_{(0)} \cdot r'(m_{(1)} \cdot r(m_{(2)})) = m_{(0)} \cdot (rr')(m_{(1)}) = m \cdot rr'.$$

Let $\{r_i, c^i\}_{i=1}^n$, $r_i \in R$, $c^i \in \mathcal{C}$ be a dual basis of \mathcal{C} as a left A -module. Notice that for all $c \in \mathcal{C}$, $\Delta_{\mathcal{C}}(c) = \sum_i r_i(c) \cdot c^i_{(1)} \otimes_A c^i_{(2)}$. On the other hand

$$\begin{aligned} \Delta_{\mathcal{C}}(c) &= \sum_i c_{(1)} \otimes_A r_i(c_{(2)}) \cdot c^i = \sum_i c_{(1)} \cdot r_i(c_{(2)}) \otimes_A c^i \\ &= \sum_{i,j} r_j(c_{(1)} \cdot r_i(c_{(2)})) \cdot c^j \otimes_A c^i = \sum_{i,j} r_i r_j(c) \cdot c^i \otimes_A c^j. \end{aligned}$$

This implies that

$$\sum_i r_i \otimes_A c^i_{(1)} \otimes_A c^i_{(2)} = \sum_{i,j} r_i r_j \otimes_A c^i \otimes_A c^j.$$

Using this equality one easily finds that given $M \in \mathbf{M}_R$, M is a right \mathcal{C} -comodule with the coaction $\rho^M(m) = \sum_i m \cdot r_i \otimes_A c^i$. Also, one easily checks that the maps described provide the required isomorphism of categories. \square

Now we can prove Theorem 4.1

Proof. (1) \Leftrightarrow (2). By Lemma 4.2, (1) implies that \mathcal{C} is a finitely generated projective left A -module. Lemma 4.3 shows that the forgetful functor is the restriction of scalars functor $\mathbf{M}_R \rightarrow \mathbf{M}_A$. By Lemma 3.1, this functor has the right adjoint $- \otimes_A \mathcal{C}$. By [20, Theorem 3.15] the restriction of scalars functor has the same left and right adjoint if and only if the extension $A \rightarrow R$ is Frobenius.

(2) \Leftrightarrow (3). Since \mathcal{C} is a finitely generated projective left A -module the map $\alpha : \mathcal{C} \rightarrow \text{Hom}_A(R, A)$, given by $\alpha(c)(r) = r(c)$ is bijective by [27, 3.5 Duality

Lemma]. Clearly α is an (A, R) -bimodule map. Thus we have $\mathcal{C} \cong \text{Hom}_A(R, A)$ as (A, R) -bimodules. The extension $A \rightarrow R$ is Frobenius iff $R \cong \text{Hom}_A(R, A)$, i.e., iff $\mathcal{C} \cong R$ as (A, R) -bimodules.

(3) \Leftrightarrow (4). This follows from the bijective correspondence $\theta : \mathcal{C}^A \rightarrow {}_A\text{Hom}_R(R, \mathcal{C})$, $\theta(e)(r) = e \cdot r = e_{(1)} \cdot r(e_{(2)})$, $\theta^{-1}(f) = f(\epsilon_{\mathcal{C}})$. \square

In the case of the canonical coring associated to a ring extension $B \rightarrow A$, Theorem 4.1 gives the criteria when the extension $A \rightarrow {}_B\text{End}(A)$ is Frobenius. The ring structure on ${}_B\text{End}(A)$ is given by the opposite composition of maps. For example if $B \rightarrow A$ is itself a Frobenius extension with a Frobenius system $E \in {}_B\text{Hom}_B(A, B)$, $a_i, \bar{a}^i \in A$, $i = 1, \dots, n$ (i.e., for all $a \in A$, $\sum_i a_i E(\bar{a}^i a) = \sum_i E(a a_i) \bar{a}^i = a$), then $A \rightarrow R$ is Frobenius by the endomorphism ring theorem [18]. In this case $e = \sum_i a_i \otimes_B \bar{a}^i$ and the inverse of ϕ is given by $\phi^{-1}(a \otimes_B a')(a'') = E(a'' a) a'$, for all $a, a', a'' \in A$. In the case of the coring associated to an entwining structure $(A, C)_{\psi}$, R becomes a generalised smashed product $C^{*op} \#_{\bar{\psi}} A$, and therefore Theorem 4.1 implies [12, Theorem 2.4] or its entwined module formulation [4, Proposition 3.5].

5. GALOIS-TYPE CORINGS

In this section we study A -corings \mathcal{C} for which $A \in \mathbf{M}^{\mathcal{C}}$.

Lemma 5.1. *For an A -coring \mathcal{C} , A is a right \mathcal{C} comodule if and only if there exists a grouplike $g \in \mathcal{C}$ (cf. [27, 1.7 Definition]).*

Proof. If $A \in \mathbf{M}^{\mathcal{C}}$ define $g = \rho^A(1) \in A \otimes_A \mathcal{C} \cong \mathcal{C}$. Then

$$\Delta_{\mathcal{C}}(g) = \Delta_{\mathcal{C}}(\rho^A(1)) = (A \otimes_A \Delta_{\mathcal{C}}) \circ \rho^A(1) = (\rho^A \otimes_A \mathcal{C}) \circ \rho^A(1) = \rho^A(1) \otimes_A \rho^A(1) = g \otimes_A g.$$

Furthermore $\epsilon_{\mathcal{C}}(g) = (A \otimes_A \epsilon_{\mathcal{C}}) \circ \rho^A(1) = 1$, so that g is a grouplike as required.

Conversely, if $g \in \mathcal{C}$ is a grouplike, define $\rho^A : A \rightarrow \mathcal{C} \cong A \otimes_A \mathcal{C}$, $a \mapsto g \cdot a = 1 \otimes_A g \cdot a$. Clearly ρ^A is a right A -module map. The fact that $\epsilon_{\mathcal{C}}$ is a right A -module map implies for all $a \in A$, $(A \otimes_A \epsilon_{\mathcal{C}}) \circ \rho^A(a) = \epsilon_{\mathcal{C}}(g) a = a$. Finally, since $\Delta_{\mathcal{C}}$ is a right A -module map we have $(A \otimes_A \Delta_{\mathcal{C}}) \circ \rho^A(a) = 1 \otimes_A \Delta_{\mathcal{C}}(g) \cdot a = 1 \otimes_A g \otimes_A g \cdot a$. On the other hand $(\rho^A \otimes_A \mathcal{C}) \circ \rho^A(a) = \rho^A(1) \otimes_A g \cdot a = 1 \otimes_A g \otimes_A g \cdot a$. Put together this implies that ρ^A is a right \mathcal{C} -coaction. \square

For example, if \mathcal{C} is the canonical A -coring associated to a ring extension $B \rightarrow A$ then $g = 1 \otimes_B 1$ is a grouplike by [27, 1.9 Proposition (a)], and hence A is a right \mathcal{C} -comodule via $a \mapsto 1 \otimes_B a$.

For the rest of this section we assume that $A \in \mathbf{M}^{\mathcal{C}}$ and denote the corresponding grouplike by g . In this case, for each $M \in \mathbf{M}^{\mathcal{C}}$ we define the *coinvariants* by

$$M^{co\mathcal{C}} = \{m \in M \mid \rho^M(m) = m \otimes_A g\}.$$

In particular, let $B = A^{co\mathcal{C}} = \{b \in A \mid b \cdot g = g \cdot b\}$, i.e., B is the centraliser of g in A . Clearly B is a subring of A , ρ^A is a (B, A) -bimodule map and $M^{co\mathcal{C}} \in \mathbf{M}_B$.

Proposition 5.2. *Let \mathcal{C} be an A -coring with a grouplike g , and let $B = A^{co\mathcal{C}}$. Then the functor $G : \mathbf{M}^{\mathcal{C}} \rightarrow \mathbf{M}_B$, $M \mapsto M^{co\mathcal{C}}$ is the right adjoint of the induction functor $- \otimes_B A : \mathbf{M}_B \rightarrow \mathbf{M}^{\mathcal{C}}$. Here, for any $M \in \mathbf{M}_B$, the right \mathcal{C} -coaction on $M \otimes_B A$ is given by $M \otimes_B \rho^A$.*

Proof. For any $M \in \mathbf{M}^{\mathcal{C}}$ define an additive map $\Psi_M : M^{co\mathcal{C}} \otimes_B A \rightarrow M$, $m \otimes_B a \mapsto m \cdot a$. Clearly Ψ_M is a right A -module map. Furthermore for any $m \in M^{co\mathcal{C}}$ and $a \in A$ we have $\rho^M(m \cdot a) = m \otimes_A g \cdot a$ as well as

$$(\Psi_M \otimes_A \mathcal{C}) \circ \rho^{G(M) \otimes_B A}(m \otimes_B a) = (\Psi_M \otimes_A \mathcal{C})(m \otimes_B 1 \otimes_A g \cdot a) = m \otimes_A g \cdot a.$$

Therefore Ψ_M is a \mathcal{C} -comodule map. One easily checks that Ψ_M is natural in M .

Next for any $N \in \mathbf{M}_B$ define an additive map $\Phi_N : N \rightarrow (N \otimes_B A)^{co\mathcal{C}}$, $n \mapsto n \otimes_B 1$. Notice that $(N \otimes_B \rho^A)(n \otimes_B 1) = n \otimes_B 1 \otimes_A g$ so that Φ_N is well-defined. One easily checks that Φ_N is a right B -module map natural in N .

Finally, with M, N as before, take any $m \in M^{co\mathcal{C}}$ and compute $\Psi_M \circ \Phi_{M^{co\mathcal{C}}}(m) = \Psi_M(m \otimes_B 1) = m$. Then take any $n \in N$, $a \in A$ and compute $\Psi_{N \otimes_B A} \circ (\Phi_N \otimes_B A)(n \otimes_B a) = \Psi_{N \otimes_B A}(n \otimes_B 1 \otimes_B a) = n \otimes_B a$. This proves that Φ and Ψ are the required adjunctions. \square

In fact Proposition 5.2 is a special case of the following Hom-Tensor relation for a coring. Let B, A be rings, \mathcal{C} an A -coring, and let V be a (B, A) -bimodule and right \mathcal{C} -comodule such that the coaction is a left B -module map. Then for any right B -module N and a right \mathcal{C} -comodule M one has

$$\mathrm{Hom}^{\mathcal{C}}(N \otimes_B V, M) \cong \mathrm{Hom}_B(N, \mathrm{Hom}^{\mathcal{C}}(V, M)).$$

Explicitly for any $f \in \text{Hom}^{\mathcal{C}}(N \otimes_B V, M)$, $f \mapsto (n \mapsto (v \mapsto f(n \otimes_B v)))$ and for any $g \in \text{Hom}_B(N, \text{Hom}^{\mathcal{C}}(V, M))$, $g \mapsto (n \otimes_B v \mapsto g(n)(v))$.

Definition 5.3. Let \mathcal{C} be an A -coring with a grouplike g , and let $B = A^{co\mathcal{C}}$. \mathcal{C} is said to be *Galois* iff there exists an A -coring isomorphism $\chi : A \otimes_B A \rightarrow \mathcal{C}$ such that $\chi(1 \otimes_B 1) = g$.

For example, if A is a division ring, any A -coring which is generated by a grouplike g as an (A, A) -bimodule is Galois (cf. [27, 2.2 Fundamental Lemma]). Our present terminology is motivated by the following two examples.

Example 5.4. Let $(A, C)_{\psi}$ be an entwining structure and let $\mathcal{C} = A \otimes C$ as in Proposition 2.2. \mathcal{C} is a Galois A -coring if and only if $(A, C)_{\psi}$ is the canonical entwining structure of a C -Galois extension $B \subset A$.

Proof. If \mathcal{C} corresponds to a C -Galois extension $B \subset A$ then $A \otimes_B A \cong A \otimes C$ as A -corings via the canonical map can , and hence \mathcal{C} is Galois. Conversely, if \mathcal{C} is Galois then A is a right \mathcal{C} -comodule, and by the correspondence in Proposition 2.2 it is an $(A, C)_{\psi}$ -module. The corresponding grouplike in \mathcal{C} is $g = \rho^A(1) = 1_{(0)} \otimes 1_{(1)}$. Furthermore $A^{co\mathcal{C}} = \{b \in A \mid \rho^A(b) = b1_{(0)} \otimes 1_{(1)}\} = A^{coC}$, since $A \in \mathbf{M}_A^C(\psi)$. For the same reason the A -coring isomorphism $\chi : A \otimes_B A \rightarrow A \otimes C$ explicitly reads $\chi(a \otimes_B a') = a \cdot (1_{(0)} \otimes 1_{(1)}) \cdot a' = a1_{(0)}a'_{\alpha} \otimes 1_{(1)}^{\alpha} = aa'_{(0)} \otimes a'_{(1)}$ and thus coincides with the canonical map can . This proves that $B \subset A$ is a C -Galois extension and by the uniqueness of the canonical entwining structure, $(A, C)_{\psi}$ must be the canonical entwining structure associated to $B \subset A$. \square

Example 5.5. Let (A, C, ψ) be a weak entwining structure corresponding to a weak C -Galois extension $B \subset A$ as described in Example 2.4. Then the corresponding A -coring $\mathcal{C} \subset A \otimes C$ given in Proposition 2.3 is Galois.

Proof. It suffices to show that $\text{Im}(can) = \mathcal{C}$, then can will provide the required isomorphism of A -corings. Notice that from the definition of ψ in Example 2.4 it follows that $\text{Im}\psi \subseteq \text{Im}(can)$. Since a typical element of \mathcal{C} is of the form $a1_{\alpha} \otimes c^{\alpha}$ and can is a left A -module map we have $a1_{\alpha} \otimes c^{\alpha} \in \text{Im}(can)$. Therefore $\mathcal{C} \subseteq \text{Im}(can)$.

On the other hand, since A is a weak entwined module we have for all $a \in A$, $\rho^A(a) = \rho^A(a1) = a_{(0)}1_\alpha \otimes a_{(1)}^\alpha \in \mathcal{C}$. In the view of the fact that $\text{can}(a \otimes_B a') = a\rho^A(a')$ this implies that $\text{Im}(\text{can}) \subseteq \mathcal{C}$. \square

Theorem 5.6. *Let \mathcal{C} be an A -coring with a grouplike g , $B = A^{\text{co}\mathcal{C}}$, and let $G : \mathbf{M}^{\mathcal{C}} \rightarrow \mathbf{M}_B$, $M \mapsto M^{\text{co}\mathcal{C}}$. If \mathcal{C} is Galois and A is a faithfully flat left B -module then the functors G and $-\otimes_B A : \mathbf{M}_B \rightarrow \mathbf{M}^{\mathcal{C}}$ are inverse equivalences. Conversely, if G and $-\otimes_B A$ are inverse equivalences then \mathcal{C} is Galois. In this case if \mathcal{C} is a flat left A -module then A is a faithfully flat left B -module.*

Proof. Assume that \mathcal{C} is Galois and A is a faithfully flat left B -module. First notice that $\chi : a \otimes_B a' \mapsto a \cdot g \cdot a'$. For all $M \in \mathbf{M}^{\mathcal{C}}$ consider the following commuting diagram of right \mathcal{C} -comodule maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & M^{\text{co}\mathcal{C}} \otimes_B A & \longrightarrow & M \otimes_B A & \longrightarrow & (M \otimes_A \mathcal{C}) \otimes_B A \\ & & \downarrow \Psi_M & & \downarrow M \otimes_A \chi & & \downarrow (M \otimes_A \mathcal{C}) \otimes_A \chi \\ 0 & \longrightarrow & M & \xrightarrow{\rho^M} & M \otimes_A \mathcal{C} & \xrightarrow{\ell_{MC}} & M \otimes_A \mathcal{C} \otimes_A \mathcal{C} \end{array}$$

The maps in the top row are the obvious inclusion and $m \otimes_B a \mapsto \rho^M(m) \otimes_B a - m \otimes_A g \otimes_B a$, while $\ell_{MC} = \rho^M \otimes_A \mathcal{C} - M \otimes_A \Delta_{\mathcal{C}}$ is the coaction equalising map. The top row is exact since it is a defining sequence of $M^{\text{co}\mathcal{C}}$ tensored with A , and $-\otimes_B A$ is exact. The bottom row is exact too. Since χ is a bijection, so are $M \otimes_A \chi$ and $(M \otimes_A \mathcal{C}) \otimes_A \chi$. Therefore Ψ_M is an isomorphism in $\mathbf{M}^{\mathcal{C}}$.

For all $N \in \mathbf{M}_B$ consider the following commutative diagram of right B -module maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & N \otimes_B A & \longrightarrow & N \otimes_B A \otimes_B A \\ & & \downarrow \Phi_N & & \downarrow = & & \downarrow N \otimes_B \chi \\ 0 & \longrightarrow & (N \otimes_B A)^{\text{co}\mathcal{C}} & \longrightarrow & N \otimes_B A & \longrightarrow & N \otimes_B A \otimes_A \mathcal{C} \end{array}$$

The maps in the top row are: $n \mapsto n \otimes_B 1$ and $n \otimes_B a \mapsto n \otimes_B a \otimes_B 1 - n \otimes_B 1 \otimes_B a$, and the top row is exact by the faithfully flat descent. The bottom row is the defining sequence of $(N \otimes_B A)^{\text{co}\mathcal{C}}$ and hence is exact. This implies that Φ_N is an isomorphism in \mathbf{M}_B and completes the proof of the fact that G and $-\otimes_B A$ are inverse equivalences.

Conversely, assume that G and $- \otimes_B A$ are inverse equivalences. Notice that $\phi : A \rightarrow \mathcal{C}^{co\mathcal{C}}$, $a \mapsto a \cdot g$ is an (A, B) -bimodule isomorphism. Since \mathcal{C} is a right \mathcal{C} -comodule via the coproduct, there is a corresponding adjunction $\Psi_{\mathcal{C}} : \mathcal{C}^{co\mathcal{C}} \otimes_B A \rightarrow \mathcal{C}$, and it is bijective. Define $\chi : A \otimes_B A \rightarrow \mathcal{C}$, $\chi = (\phi \otimes_B A) \circ \Psi_{\mathcal{C}}$. Explicitly $\chi : a \otimes_B a' \mapsto a \cdot g \cdot a'$. Clearly χ is an (A, A) -bimodule map such that $\chi(1 \otimes_B 1) = g$. Furthermore since g is a grouplike and $\Delta_{\mathcal{C}}$ is an (A, A) -bimodule map we have $\Delta_{\mathcal{C}} \circ \chi(a \otimes_B a') = a \cdot g \otimes_A g \cdot a'$, for all $a, a' \in A$. On the other hand

$$(\chi \otimes_A \chi) \circ \Delta_{A \otimes_B A}(a \otimes_B a') = \chi(a \otimes_B 1) \otimes_A \chi(1 \otimes_B a') = a \cdot g \otimes_A g \cdot a'.$$

Therefore χ is an A -coring isomorphism and hence \mathcal{C} is Galois.

If \mathcal{C} is a flat left A -module then both kernels and cokernels of any morphism in $\mathbf{M}^{\mathcal{C}}$ are right \mathcal{C} -comodules (i.e., $\mathbf{M}^{\mathcal{C}}$ is an Abelian category). Therefore any sequence of \mathcal{C} -comodule maps is exact if and only if it is exact as a sequence of additive maps. Since $- \otimes_B A$ is an equivalence, it preserves and reflects exact sequences. By the above observation it does so even as viewed as a functor from \mathbf{M}_B to the category of \mathbf{Z} -modules. Therefore A is a faithfully flat left B -module. \square

In view of Example 5.4, Theorem 5.6 is a generalisation of [5, Corollary 3.11] which in turn is a generalisation of [26, Theorem 3.7] (from which the idea of the proof is taken).

6. COMMENTS ON DUALITY AND OUTLOOK

Throughout this section we assume that k is a field. The results of the previous sections have dual counterparts. To introduce them we first propose the following dualisation of the notion of a coring.

Let C be a coalgebra. A C -algebroid (or a C -ring) is a (C, C) -bicomodule \mathcal{A} together with (C, C) -bicomodule maps $\mu_{\mathcal{A}} : \mathcal{A} \square_C \mathcal{A} \rightarrow \mathcal{A}$ and $\eta_{\mathcal{A}} : C \rightarrow \mathcal{A}$ such that

$$\mu_{\mathcal{A}} \circ (\mu_{\mathcal{A}} \square_C \mathcal{A}) = \mu_{\mathcal{A}} \circ (\mathcal{A} \square_C \mu_{\mathcal{A}}), \quad \mu_{\mathcal{A}} \circ (\eta_{\mathcal{A}} \square_C \mathcal{A}) \circ {}^A\rho = \mu_{\mathcal{A}} \circ (\mathcal{A} \square_C \eta_{\mathcal{A}}) \circ \rho^A = \mathcal{A}.$$

Here \square_C denotes the cotensor product of (C, C) -bicomodules, and ${}^A\rho, \rho^A$ are left, right coactions of C on \mathcal{A} . The map $\mu_{\mathcal{A}}$ is called a product and $\nu_{\mathcal{A}}$ is called a unit

of the C -algebroid \mathcal{A} . Obviously a k -algebroid is simply a k -algebra. An example of a C -algebroid can be obtained by dualisation of Example 2.1.

Example 6.1. Let $\pi : C \rightarrow B$ be a morphism of coalgebras, then C is a (B, B) -comodule via $(C \otimes \pi) \circ \Delta$ and $(\pi \otimes C) \circ \Delta$. Define $\mathcal{A} = C \square_B C$. Then \mathcal{A} is a C -algebroid with product $\mu_{\mathcal{A}} : A \square_C \mathcal{A} \cong C \square_B C \square_B C \rightarrow \mathcal{A}$, $\mu_{\mathcal{A}} = C \square_B \epsilon \square_B C$ and unit $\nu_{\mathcal{A}} = \Delta$.

A right \mathcal{A} module is a right C -comodule M with a map $\rho_M : M \square_C \mathcal{A} \rightarrow M$ such that

$$\rho_M \circ (\rho_M \square_C \mathcal{A}) = \rho_M \circ (M \square_C \mu_{\mathcal{A}}), \quad \rho_M \circ (M \square_C \eta_{\mathcal{A}}) \circ \rho^M = M,$$

where ρ^M is the right coaction of C on M . The map ρ_M is called the action of \mathcal{A} on M . A map of \mathcal{A} -modules is a right C -colinear map which respects the actions. The category of right \mathcal{A} -modules is denoted by $\mathbf{M}_{\mathcal{A}}$.

Dualising Proposition 2.2 one obtains

Proposition 6.2. *For an entwining structure $(A, C)_{\psi}$, view $\mathcal{A} = C \otimes A$ as a (C, C) -bicomodule with the left coaction ${}^A\rho = \Delta \otimes C$ and the right coaction $\rho^A = (C \otimes \psi) \circ (\Delta \otimes A)$. Then \mathcal{A} is a C -algebroid with the product $\mu_{\mathcal{A}} : \mathcal{A} \square_C \mathcal{A} \cong C \otimes A \otimes A \rightarrow \mathcal{A}$, $\mu_{\mathcal{A}} = C \otimes \mu$, and the counit $\eta_{\mathcal{A}} = C \otimes 1$.*

Conversely if $C \otimes A$ is a C -algebroid with the product and the unit above and the natural left C -comodule structure $\Delta \otimes A$, then $(A, C)_{\psi}$ is an entwining structure, where $\psi = (\epsilon \otimes A \otimes C) \circ \rho^{C \otimes A}$.

Under this bijective correspondence $\mathbf{M}_{\mathcal{A}} = \mathbf{M}_A^C(\psi)$.

One can also dualise main results in Section 3. For example, given a right C -comodule M one defines a right \mathcal{A} -module structure on $M \square_C \mathcal{A}$ via $M \square_C \mu_{\mathcal{A}}$. Dualising Lemma 3.1 one easily finds that the functor $-\square_C \mathcal{A} : \mathbf{M}^C \rightarrow \mathbf{M}_{\mathcal{A}}$ is left adjoint of the forgetful functor $\mathbf{M}_{\mathcal{A}} \rightarrow \mathbf{M}^C$. One can then study separability of these functors and thus obtain results dual to Theorem 3.3 and Theorem 3.5:

Theorem 6.3. *The functor $-\square_C \mathcal{A} : \mathbf{M}^C \rightarrow \mathbf{M}_{\mathcal{A}}$ is separable if and only if there exists a map $e : \mathcal{A} \rightarrow k$ such that $(e \otimes C) \circ \rho^A = (C \otimes e) \circ {}^A\rho$ and $e \circ \eta_{\mathcal{A}} = \epsilon$.*

The forgetful functor $\mathbf{M}_{\mathcal{A}} \rightarrow \mathbf{M}^C$ is separable if and only if there exists $\gamma \in {}^C\mathrm{Hom}^C(C, \mathcal{A} \square_C \mathcal{A})$ such that

$$(\mu_{\mathcal{A}} \square_C \mathcal{A}) \circ (\mathcal{A} \square_C \gamma) \circ \rho^{\mathcal{A}} = (\mathcal{A} \square_C \mu_{\mathcal{A}}) \circ (\gamma \square_C \mathcal{A}) \circ {}^{\mathcal{A}}\rho.$$

and $\mu_{\mathcal{A}} \circ \gamma = \eta_{\mathcal{A}}$.

Finally one can proceed to formulate results dual to ones described in Sections 4 and 5. In particular one easily finds that C is an \mathcal{A} -module provided there is a nontrivial character $\kappa : \mathcal{A} \rightarrow k$. Using the natural identification $C \square_C \mathcal{A} \cong \mathcal{A}$ the action of \mathcal{A} on C is a map $\rho_C : \mathcal{A} \rightarrow C$, $a \mapsto \kappa(a_{(0)})a_{(1)}$. In this case one can study invariants of C , $I = \mathrm{span}\{\kappa(a_{(0)})a_{(1)} - a_{(-1)}\kappa(a_{(0)}) \mid a \in \mathcal{A}\}$, where ${}^{\mathcal{A}}\rho(a) = a_{(-1)} \otimes a_{(0)}$ denotes the left coaction of C on \mathcal{A} , and find that I is a coideal. Hence $B = C/I$ is a coalgebra. One can proceed to define a *Galois C -algebroid* as a C -algebroid \mathcal{A} with a nontrivial character κ such that there is an isomorphism of C -algebroids $\chi : \mathcal{A} \rightarrow C \square_B C$ such that $\kappa = (\epsilon \square_B \epsilon) \circ \chi$. The details of all the results quoted here as well as derivation of results concerning functor $-\square_B \mathcal{A} : \mathbf{M}^B \rightarrow \mathbf{M}_{\mathcal{A}}$ are left to the reader.

In this paper we have shown that corings provide a natural framework for studying (weak) entwining structures and (weak) entwined modules, and that some general results for the latter can be deduced from the corresponding results for corings. It seems natural and, indeed, desired to study other properties of entwining structures from the point of view presented in this paper. On the other hand already known properties of entwining structures might suggest similar properties for corings. For example, we expect that the pairs of adjoint functors considered in this paper are in fact special cases of pairs of induction functors that involve both a coring and an algebroid. Another line of development is suggested by the following observation made by S. Caenepeel [8]. The relationship between corings and weak entwining structures described in Proposition 2.3 can be made more complete along the lines similar to Proposition 2.2 provided one introduces the following generalisation of a coring. For a ring A , an *A -pre-coring* is an (A, A) -bimodule \mathcal{C} non-unital as a right A -module (i.e., it is not required that $c \cdot 1 = c$ for all $c \in \mathcal{C}$) with (A, A) -bimodule maps $\Delta_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C}$ and $\epsilon_{\mathcal{C}} : \mathcal{C} \rightarrow A$ satisfying the same axioms as

for a coring and such that $c \cdot 1 = \epsilon_{\mathcal{C}}(c_{(1)} \cdot 1)c_{(2)}$ for all $c \in \mathcal{C}$. S. Caenepeel then proves that given an A -pre-coring \mathcal{C} , the map $p : \mathcal{C} \rightarrow \mathcal{C}$ given by $p(c) = c \cdot 1$ is an (A, A) -bilinear projection, $p \circ p = p$ such that $(p \otimes_A p) \circ \Delta_{\mathcal{C}} = \Delta_{\mathcal{C}} \circ p$. Furthermore $\overline{\mathcal{C}} = \text{Coim}(p) \cong \underline{\mathcal{C}} = \text{Im}(p)$ is an A -coring. (A, C, ψ) is a weak entwining structure if and only if $A \otimes C$ is a pre-coring with structure maps defined as in Proposition 2.2. Proposition 2.3 can be deduced from this result. Since pre-corings appear naturally in the context of weak entwining structures it would be interesting to study general properties of pre-corings along the lines of the present paper or of [27]. These are the topics for further work.

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